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§0 Last time

Recall: A function $f: X \to Y$ where $(X, \leq)$ and $(Y, \preceq)$ are posets is:

- monotone: $x_1 \leq x_2 \Rightarrow f(x_1) \preceq f(x_2)$,
- order preserving: $f(x_1) \preceq f(x_2) \Rightarrow x_1 \leq x_2$
- embedding: monotone + order preserving.
- isomorphism: surjective embedding

$(X, \leq)$ and $(Y, \preceq)$ are isomorphic means they're the same up to relabeling

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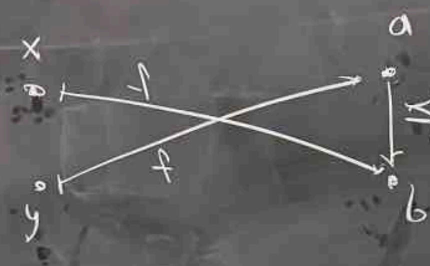
$(\mathbb{N}, |) \xrightarrow{id} (\mathbb{N}, \leq)$

Example:

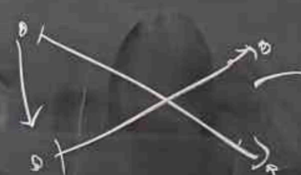


$\rightsquigarrow$ is monotone but not order preserving

$f(x) = b \geq a = f(y)$

but $x \ngeq y$

Example:



$\rightsquigarrow$ is order preserving but not monotone

$(\mathbb{N}, \leq) \xrightarrow{id} (\mathbb{N}, |)$

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§ 1 Isomorphic and Some examples

Notation: A map $f:(X,\leq)\longrightarrow(Y,\preceq)$ is just a monotone function $f:X\longrightarrow Y$.

$f: X \to Y$

$g: Z \to W$

Prop: Let $f:(X,\leq)\to(Y,\preceq)$ be a map. TFAE.

(1) f is an isomorphism

(2) $\exists\ g:(Y,\preceq)\to(X,\leq)$ s.t. $f\circ g=id_Y$ and $g\circ f=id_X$.

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(2) $\Rightarrow$ (1) Need to check that f is an surj. embedding, i.e., f is monotone, order preserving and surj. But, note that $g : Y \to X$ is an inverse function to f $\Rightarrow$ f is bij $\Rightarrow$ f is surj. Assume that $f(x_1) \preceq f(x_2) \Longrightarrow \underbrace{g(f(x_1))}_{x_1} \le g(f(x_2)) = x_2$, so f is order preserving. $\boxtimes$

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Example: $\mathbb{N} \neq \mathbb{Q}$ but $\# \mathbb{N} = \# \mathbb{Q}$.

Show that $(\mathbb{N}, \leq)$ and $(\mathbb{Q}, \leq)$ are not isomorphic.

Def'n: $(X, \leq)$ and $(Y, \preceq)$ are isomorphic written $(X, \leq) \simeq (Y, \preceq)$ if $\exists$ an isomorphism $(X, \leq) \to (Y, \preceq)$.

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Exercise: Show $\simeq$ is an equivalence relation on posets.

Pf of Example: Suppose $f: (\mathbb{N}, \leq) \to (\mathbb{Q}, \leq)$ is an isom.

Observe that as f is monotone and inj, $f(1) < f(2)$.

So, $\exists$ $x \in \mathbb{Q}$ s.t. $f(1) < x < f(2)$ (e.g. $x = \frac{f(1)+f(2)}{2}$)

A f^{-1} is monotone and inj $\Rightarrow$ $\underbrace{f^{-1}(f(1))}_{1} < f^{-1}(x) < f^{-1}(f(2)) = 2$

Contradiction $\blacksquare$

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e.g. $(\mathbb{N}, \leq) \not\cong (\mathbb{Z}, \leq)$ b/c

$(\mathbb{Z}, \leq)$ has no least element but $(\mathbb{N}, \leq)$ does.

Exercise: Formalize this

Thm: For any poset $(X, \leq)$ $\exists$ an embedding

$F : (X, \leq) \to (P(S)^{\,}, \subseteq)$

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Pf: Set $S = X$, define

$$F : X \longrightarrow P(X)$$

$$x \longmapsto A_x = \{y \in X : y \leq x\}.$$

Monotone: Assume that $x_1 \leq x_2$. Let $y \in A_{x_1}$. So, $y \leq x_1 \Longrightarrow y \leq x_2$. So, $y \in A_{x_2}$. Thus,

$$F(x_1) = A_{x_1} \subseteq A_{x_2} = F(x_2)$$ ✓

Order preserving: Assume that $A_{x_1} = F(x_1) \subseteq F(x_2) = A_{x_2}$. Note, $x_1 \in A_{x_1}$. So $x_1 \in A_{x_2} \Longrightarrow x_1 \leq x_2$ $\blacksquare$

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OBS: $(X, \leq) = \bullet \to \bullet \to \bullet$ is never isomorphic to $(P(S), \subseteq)$ for any set S.

Pf: Assume that $(X, \leq) \simeq (P(S), \subseteq)$. In particular $\# X = \# P(S)$. But, X is finite and $\# S < \# P(S)$. Clearly S is finite. So if $\# S = n$, then $\# P(S) = 2^n$. So, $3 = 2^n$. Contradiction. $\boxtimes$

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Def'n: Let S be a set and $P_{fin}(S)$ for the set of finite subsets of S.

Def'n: A number $n \in \mathbb{N}$ is square free if $m^2 \nmid n$ for any m.

Thm: $\left(\{\text{Positive square-free numbers}\}, |\right) \cong \left(P_{fin}(\mathbb{N}), \subseteq\right)$.

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Lemma 1: $\left(\left\{ \begin{matrix} \text{pos. sq. free} \\ \text{numbers} \end{matrix} \right\}, \mid \right) \cong \left(\mathcal{P}_{fin}(\mathbb{P}), \subseteq \right)$

Pf: Define $F : \left\{ \begin{matrix} \text{pos. sq. free} \\ \geq 2 \text{ numbers} \end{matrix} \right\} \longrightarrow \mathcal{P}_{fin}(\mathbb{P})$

$$n \longmapsto \{ p \in \mathbb{P} : p \mid n \}$$

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Obs: $2 \leq n$ is sq.-free $\iff$ $n = p_1 \cdots p_\ell$ w/ $p_1 < \cdots < p_\ell$.

pf: Exercise.

Define $G : \mathcal{P}_{fin}(\mathbb{P}) \longrightarrow \left\{ \begin{array}{c} \text{Sq.-free} \\ \text{numbers} \geq 2 \end{array} \right\}$

$$\{p_1, \ldots, p_\ell\} \longmapsto p_1 \cdots p_\ell$$

1.4.1 Observe that by obs. $F \circ G$ and $G \circ F$ are identity.

<u>F monotone</u>: If $n = p_1 \cdots p_\ell \mid m = q_1 \cdots q_r$ then

by FTA $\Rightarrow$ $\underset{F(n)}{\underset{\parallel}{\{p_1, \dots, p_\ell\}}} \subseteq \underset{F(m)}{\underset{\parallel}{\{q_1, \dots, q_\ell)}}$

<u>G monotone</u>: Assume $\{p_1, \dots, p_\ell\} \subseteq \{q_1, \dots, q_n\}$

then $G(\{p_1, \dots, p_\ell\}) = p_1 \cdots p_\ell \mid q_1 \cdots q_r = G(\{q_1, \dots, q_n\})$ $\blacksquare$

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Lemma 2: If $\#X = \#Y$, then

$$(\mathcal{P}_{fin}(X), \subseteq) \simeq (\mathcal{P}_{fin}(Y), \subseteq)$$

Pf of Thm:

$$(\{\text{sq. free numbers}\}, |) \underset{\text{Lem 1}}{\simeq} (\mathcal{P}_{fin}(\mathbb{P}), \subseteq) \underset{\text{Lem 2}}{\simeq} (\mathcal{P}_{fin}(\mathbb{N}), \subseteq) \quad \square$$

§2 Dense linear orders

<u>Def'n:</u> A poset $(X, \leq)$ is

- <u>linear</u> if $\forall x, y \in X$ either $x \leq y$ or $y \leq x$
- <u>dense</u> if $\forall x, y \in X$ w/ $x < y$ $\exists$ $z \in X$ w/ $x < z < y$.
- <u>endless</u> if $\forall x \in X$, $\exists$ $y, z \in X$ w/ $y < x < z$.

e.g.) $(\mathbb{Q} \cap [0,1], \leq)$ is linear and dense but not endless.

e.g.) $(\mathbb{Q}^2, \leq_{lex})$ $\quad (a,b) \leq (c,d) \iff \begin{cases} a<c & \text{if } a \neq c \\ b \leq d & \text{if } a=c \end{cases}$

is dense and endless but not linear.

e.g.) $(\mathbb{Z}, \leq)$ are linear and endless but not dense.

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<u>Thm</u> (Cantor): $(\mathbb{Q}, \leq)$ up to isomorphism is the unique countable linear, dense, endless poset.